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Study of laplace transform and its application in mathematical modeling

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Abstract

Laplace Transform is an effective mathematical tool used to solve complex differential equations with boundary values without finding general solutions and the values of arbitrary constants. In this research paper, the Laplace Transform is used to solve problems formulated in mathematical modeling and is demonstrated through examples.

Keywords: Laplace Transform, Inverse Laplace Transform, Time, Population, Radioactive Element.

Introduction

Population growth at any time t can be represented by an ordinary differential equation. If x(t) is the population at any time t then the rate of growth of the population

$\frac{dx}{dt}$ a x

$\Rightarrow \frac{dx}{dt} = kx$ ....(1)

Where k is the proportionality constant it is positive

Laplace Transform: Laplace transform is an operator that transforms a function of the time domain f(t) into the frequency domain. If f(t) is the function of t then the Laplace Transform of f(t) is denoted by L[f(t)] and defined as

$L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s) \quad ,t \geq 0$

The sufficient condition for a Laplace Transform to exist is that f(t) should be piece wise continuous and of exponential order.

Properties of Laplace Transform:

Linearity Property: If $f_1(t)$ and $f_2(t)$ are the two functions of t then $L[af_1(t) + bf_2(t)] = a L[f_1(t)] + b L[f_2(t)]$, Where a and b are constants.

Laplace Transform of Standard Functions

Sn.	f(t)	L[f(t)]
1.	1	1/s
2.	e^{at}	1/s-a
3.	e^{-at}	1/s+a
4.	Sin at	a/s^2 +a^2
5.	Cos at	a/s^2 +a^2
6.	Sin hat	a/s^2 +a^2
7.	Cos hat	a/s^2 +a^2
8.	t^n	$n!/s^{n+1}$

Laplace Transform of Derivatives: If L[f(t)] = F(s) then (i)L [f'(t)] = s L[f(t)] - f(0)

(ii) L [f''(t)] = s^2 L[f(t)] - sf(0) -f'(0)

Inverse Laplace Transform: If L[f(t)] = F(s) then f(t) is called inverse Laplace Transform of F(s) and it is denoted by L^{-1}{F(s)] = f(t)

Inverse Laplace Transform of Standard Functions:

Sn.	L^{-1}[F(s)]	f(t)
1.	1/s	1
2.	1/s-a	e^{at}
3.	1/s+a	e^{-at}
4.	a/s^2 +a^2	Sin at
5.	a/s^2 +a^2	Cos at
6.	a/s^2 +a^2	Sin hat
7.	a/s^2 +a^2	Cos hat
8.	$n!/s^{n+1}$	t^n

Applications: Now we demonstrate the effectiveness of Laplace Transform in Mathematical Modeling through examples.

Population growth problems

If p(t) is the population at any time t and x_0 is the initial population element at t=0 , then the rate of growth of population with respect to time is directly proportional to the current.

population i.e. $\frac{dp}{dt}$ a p

Or p'(t) = kp(t) .... (2)

Taking Laplace Transform of eq^n(1) on both sides L[p'(t)] = L[kp(t)]

⇒ sL[p(t) - p(0) = kL[p(t)]

⇒ sL[p(t)] - x_0 = kL[p(t)]

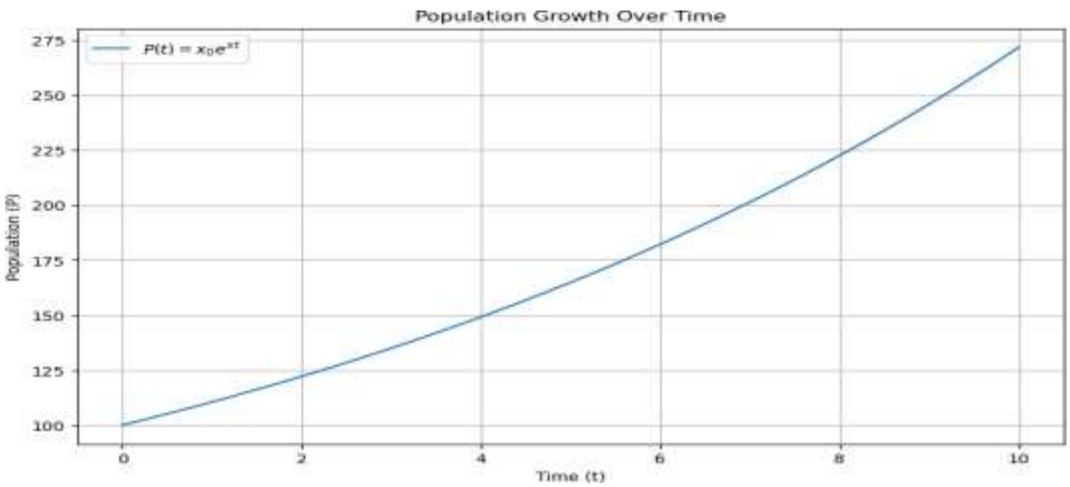
⇒(s -k)L[p(t)] = x_0

⇒ L[p(t)] = x_0/(s -k)...(3)

Taking inverse Laplace Transform of eq^n(3) on both sides p(t) = L^{-1}[x_0 /(s − k)]

⇒ p(t) = x_0e^{kt} .... (4)

Graphically it can be represented as



Graph shows that population increases exponentially with respect to time.Now we demonstrate above through an example

1.If the population of a country doubles in 50 years. In how many years will it treble assuming that the rate of increase of population is proportional to the number of inhabitants.

Sol. Let p(t) is the population at any time t (in years) And x0 is the initial population at t = 0

then by law $\frac{dp}{dt} = kp$ ....(i)

Taking Laplace Transform on both sides of above we get p(t) = $x0e^{kt}$ ....(ii)

Given at t = 50 , p(t) = 2x0 putting above in equation (ii)

$2x0 = x0e^{50k}$

⇒ $2 = e^{50k}$

⇒ 50k = log2

⇒ k = log2/50, putting above value in equation (ii)

⇒p(t) = $x0e^{(lo2/50)t}$............ (iii)

Equation (iii) gives the population at any time t Now to find out the year such that population trebles of initial population i.e. p(t) =3x0 putting above value in equation (iii) we get

$3x0 = x0e^{(log2/50)t}$

⇒$3 = e^{(log2/50)t}$

⇒ (log2/50) t = log3

⇒ t = (50/ log2) log3

⇒ At time t = (50/ log2) log3, population trebles of initial population.

Radioactive decay problems

The rate of decay of any radioactive element is proportional to the radioactive element present. Let n(t) be the radioactive element at any time t and n0 is the initial radioactive element at t=0, then the amount of radioactive element decreases at a rate proportional to the radioactive element present i.e.

$$\frac{dn}{dt} \propto n$$

i.e. $\frac{dn}{dt} = -\lambda n$ .... (5)

where λ is negative proportionality constant Taking Laplace Transform of eq^n(4) on both sides $L[n'(t)] = L[-\lambda n(t)]$

$\Rightarrow sL[n(t)] - n(0) = -\lambda L[n(t)]$

$\Rightarrow sL[n(t)] - n_0 = -\lambda L[n(t)]$

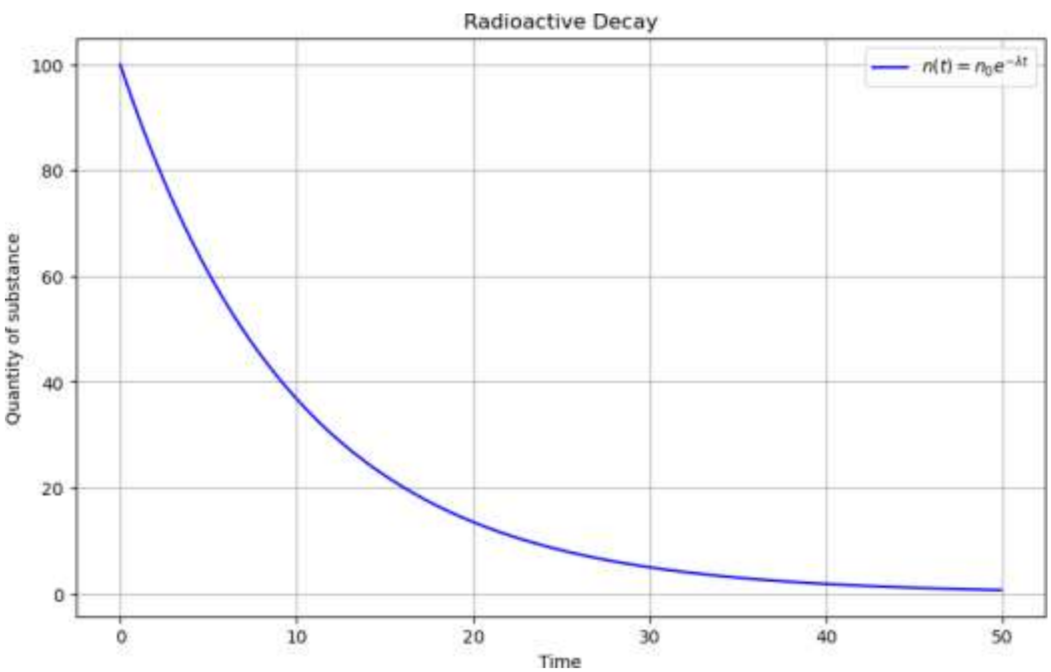
$\Rightarrow (s + \lambda)L[n(t)] = n_0$

$\Rightarrow L[n(t)] = n_0/(s + \lambda)$ .... (6)

Taking inverse Laplace Transform of eq^n(6) on both sides $n(t) = L^{-1}[n_0 /(s + \lambda)]$

$\Rightarrow n(t) = n_0 e^{-\lambda t}$ .... (7)

Graphically it can be represented as



Now we demonstrate above through an example

Problem: A certain radioactive material is known to decay at rate proportional to the amount present. If initially 600mg of the materials is present after 4 years 40 percent of the original mass has decayed. Find expression of mass at any time t.

Sol. Let n(t) is the radioactive substance at any time t (in years)

And n_0 is the initial radioactive substance at t = 0

then by law $\frac{dn}{dt} = -\lambda n$..(iv)

From eq^n (6)

we get $n(t) = n_0 e^{-\lambda t}$ ....(v)

Given that after 4 years amount of radioactive element = $n_0 - n_0 *(40/100) = 3n_0/5$

Therefore from eq^n (v) $3n_0/5 = n_0 e^{-4\lambda}$

$\Rightarrow 3/5 = e^{-4\lambda}$

$\Rightarrow 5/3 = e^{4\lambda}$

$\Rightarrow (1/4)\log(5/3) = \lambda$

$\Rightarrow 0.1277 = \lambda$

Putting above values of λ and initial value of radioactive element n_0 = 600 in eq^n (v) we the expression for radioactive element at any time t

i.e. $n(t) = 600e^{-0.1277t}$ mg ....(vi)

Conclusion

This paper discusses how the Laplace transform is used to solve problems in mathematical modeling. It has been shown that the problems of population growth and decay of radioactive elements can be formulated in the form of differential equations and can be solved easily using Laplace transform. The method is also demonstrated through examples The proposed scheme has wide application in Science, Life Science, Mathematics & Engineering.

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